THE EXISTENCE-UNIQUENESS THEOREM FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

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https://intuitiveexplanations.com/picard-lindelof-theorem/

This document is a proof of the existence-uniqueness theorem for first-order differential equations, also known as the Picard-Lindelöf or Cauchy-Lipschitz theorem. It was written with special attention to both rigor and clarity. The proof is primarily based on the one given in the textbook I used in my Differential Equations class (*Differential Equations and Their Applications* by Martin Braun, fourth edition).

1. Goal

Let f be a function of two variables and let t_0 and y_0 be real numbers. This defines an initial-value problem:

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(t_0) &= y_0. \end{aligned} \tag{1}$$

Our goal is to prove that under certain conditions, there is exactly one solution y to this differential equation. That is, we would like to prove both the *existence* and *uniqueness* of solutions to the equation.

2. Initial Steps

We will start our proof by transforming the differential equation (1) into a more convenient form. This is done by integrating both sides from t_0 to t:

$$\int_{t_0}^{t} y'(\tau)\, d\tau = \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau.$$

Here, to avoid ambiguity, we are using the variable τ as our variable of integration instead of t. The above equation reduces to

$$y(t) - y_0 = \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau,$$

since $y(t_0) = y_0$, and solving for $y(t)$ gives the equation

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau. \tag{2}$$

By this reasoning, any function satisfying (1) must also satisfy equation (2). However, the converse statement requires a little more work. Suppose that a function y satisfies equation (2). Then it follows immediately that $y(t_0) = y_0$, because

$$y(t_0) = y_0 + \int_{t_0}^{t_0} f(\tau, y(\tau))\, d\tau = y_0.$$

Now, the fundamental theorem of calculus tells us that if a function f is continuous, then $\int_a^b f(x)\, dx$ is differentiable with respect to b, and

$$\frac{d}{db} \int_a^b f(x)\, dx = f(b).$$

Hence, if we assume that if both f and y are continuous, then $f(\tau, y(\tau))$ is a continuous function of τ and we have

$$y'(t) = \frac{d}{dt} \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau = f(t, y(t)).$$

Therefore, if y is a continuous function satisfying (2) and f is continuous, then y is a solution of the original differential equation (1).

3. Outline

Our proof will consist of the following major steps:

(a) Construct a sequence of functions $\{y_0(t), y_1(t), \ldots\}$, called Picard iterates, which approximate a solution to the equation (2).
(b) Show that the sequence of functions converges and define $y(t) = \lim_{n\to\infty} y_n(t)$.
(c) Show that the function $y(t)$ satisfies equation (2).
(d) Show that the function $y(t)$ is continuous.
(e) Show that there can only be one solution to the equation (2).

Having completed these tasks, our reasoning above will imply that the function $y(t)$ is the unique solution to (1).

4. Construction of the Picard iterates

As our first approximation to a solution to the differential equation (1), we will choose the simplest possible function that satisfies the condition $y(t_0) = y_0$; that is,

$$y_0(t) = y_0.$$

Our procedure for generating better approximations is motivated by the relation (2) which is satisfied by any solution y to (1), reprinted here:

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau.$$

In particular, we will define

$$y_n(t) = y_0 + \int_{t_0}^{t} f(\tau, y_{n-1}(\tau))\, d\tau \tag{3}$$

for every $n \geq 1$. Observe that we have $y_n(t_0) = y_0$ for every $n \geq 0$, so every Picard iterate obeys the initial condition.

As a concrete example of this iteration process, consider the differential equation

$$\begin{aligned} y'(t) &= y(t) \\ y(0) &= 1, \end{aligned}$$

whose unique solution is $y(t) = e^t$. For this equation, we have $f(t, y) = y$, $t_0 = 0$, and $y_0 = 1$. This transforms the recurrence relation (3) to

$$y_n(t) = 1 + \int_0^t y_{n-1}(\tau)\, d\tau;$$

therefore,

$$\begin{aligned} y_0(t) &= 1 \\ y_1(t) &= 1 + \int_0^t d\tau = 1 + t \\ y_2(t) &= 1 + \int_0^t 1 + \tau\, d\tau = 1 + t + \frac{t^2}{2} \\ &\vdots \\ y_n(t) &= 1 + t + \frac{t^2}{2} + \cdots + \frac{t^n}{n!}. \end{aligned}$$

The astute reader will recognize $y_n(t)$ as the nth partial sum of the Maclaurin series for e^t. It follows easily, then, that the sequence $\{y_0(t), y_1(t), \ldots\}$ converges and that the limiting function $y(t) = \lim_{n\to\infty} y_n(t) = e^t$ is continuous and satisfies (2). We now show that this conclusion is true in general under suitable assumptions.

5. Bounding the Picard iterates

In general, even if the differential equation (1) has a unique solution, the solution may only be valid on a specified interval—typically because $f(t, y(t))$ is not defined for one or more values of t. Therefore, we will have to restrict our reasoning to a limited interval containing t_0. However, it is difficult to reason about the largest possible interval—that is, the largest interval over which the differential equation has a solution. Instead, we will pick a smaller interval in such a way that the behavior of the Picard iterates is easy to analyze over the interval.

To construct this interval, we will start by picking two arbitrary positive real numbers a and b. These numbers define a rectangle in the t-y plane that has vertices at $(t_0, y_0 - b)$, $(t_0, y_0 + b)$, $(t_0 + a, y_0 - b)$, and $(t_0 + a, y_0 + b)$. This rectangle is illustrated in Figure 1.

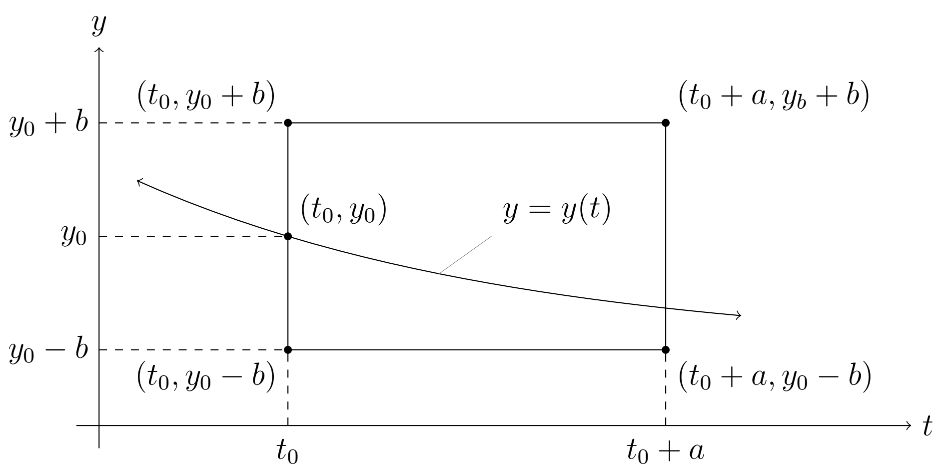

Figure 1

Let R denote the rectangle and its interior, i.e. the set of all points (t, y) such that $t_0 \le t \le t_0 + a$ and $y_0 - b \le y \le y_0 + b$. Since we are assuming that f is continuous, it follows that $|f|$ is continuous and has a maximum value on R. We let M denote this maximum value, i.e.

$$M = \max_{(t,y)\in R} |f(t, y)|.$$

Next, we consider the lines through the point (t_0, y_0) that have slope M and $-M$, respectively. These lines have equations $y = y_0 \pm M(t - t_0)$, and are shown in Figure 2.

From the figure, it is easy to see that depending on the value of M, the lines will leave the rectangle at either $t = a$ or $t = b/M$, whichever is smaller. We will denote this t-value by α, i.e.

$$\alpha = \min\left(a, \frac{b}{M}\right).$$

We will now prove that for $t_0 \le t \le t_0 + \alpha$, every Picard iterate lies between the two lines. That is, until the lines leave the rectangle R, every $y_n(t)$ lies within the shaded regions in Figure 2. We can reformulate this hypothesis as follows:

$$\begin{aligned} y_0 - M(t - t_0) \le y_n(t) \le y_0 + M(t - t_0) \\ -M(t - t_0) \le y_n(t) - y_0 \le M(t - t_0) \end{aligned}$$

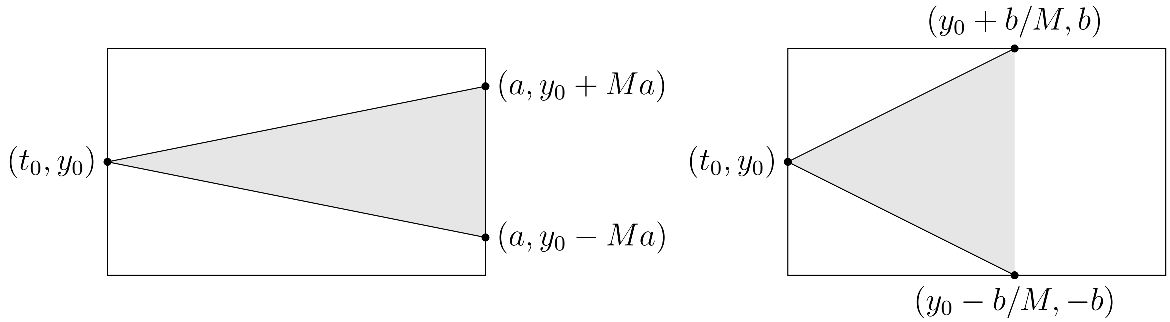

FIGURE 2

$$|y_n(t) - y_0| \leq M(t - t_0).$$

Because $M \geq 0$ and $t \geq t_0$, we need no absolute value bars on the right-hand side.

To prove the hypothesis, we use induction on n. The case of $n = 0$ follows immediately, as

$$|y_0(t) - y_0| = |y_0 - y_0| = 0 \leq M(t - t_0).$$

For the inductive case, we assume—for some $n \geq 0$—that $|y_n(t) - y_0| \leq M(t - t_0)$ for $t_0 \leq t \leq \alpha$ and seek to prove that $|y_{n+1}(t) - y_0| \leq M(t - t_0)$ on the same interval. We now use the definition (3) of the Picard iterates, reprinted here:

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau.$$

In particular, we note that

$$|y_{n+1}(t) - y_0| = \left| \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau \right|.$$

Next, we use the following two elementary properties of definite integrals:

$$\left| \int_a^b f(x)\, dx \right| \leq \int_a^b |f(x)|\, dx$$

$$\int_a^b f(x)|g(x)|\, dx \leq \left(\max_{a \leq x \leq b} f(x) \right) \int_a^b |g(x)|\, dx.$$

Note that we have $t_0 \leq \tau \leq t \leq t_0 + \alpha$. Consequently, it follows from the inductive hypothesis that $y_n(\tau)$ lies between the lines and hence within R on this interval. Thus, $(\tau, y_n(\tau))$ lies within R for all τ from t_0 to t, and $|f(\tau, y_n(\tau))| \leq M$ over this interval. From these properties, we have:

$$\begin{aligned} \left| \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau \right| &\leq \int_{t_0}^{t} |f(\tau, y_n(\tau))|\, d\tau \\ &\leq M(t - t_0). \end{aligned}$$

We have therefore shown that $|y_{n+1}(t) - y_0| \leq M(t - t_0)$, which completes the proof that $|y_n(t) - y_0| \leq M(t - t_0)$ for every $n \geq 0$.

6. PROOF THAT THE PICARD ITERATES CONVERGE

Now that we have obtained a bound on the size of $y_n(t)$ on a suitable interval, we can show that the sequence $\{y_0(t), y_1(t), \ldots\}$ converges on that interval. We do this by rewriting $y_n(t)$ as a telescoping series:

$$y_n(t) = y_0(t) + [y_1(t) - y_0(t)] + [y_2(t) - y_1(t)] + \cdots + [y_n(t) - y_{n-1}(t)]$$

$$= y_0(t) + \sum_{k=1}^{n} \Big[y_k(t) - y_{k-1}(t) \Big],$$

so that

$$\lim_{n\to\infty} y_n(t) = y_0(t) + \sum_{k=1}^{\infty} \Big[y_k(t) - y_{k-1}(t) \Big].$$

If the infinite series $\sum_{k=1}^{\infty}[y_k(t)-y_{k-1}(t)]$ converges, then so does the sequence $\{y_0(t), y_1(t), \ldots\}$. Now, if we replace every term of a series with its absolute value and it still converges, then certainly the original series must also converge. Thus, it suffices to show the convergence of $\sum_{k=1}^{\infty} |y_k(t) - y_{k-1}(t)|$. To do this, we will use a series of approximations involving the quantity $|y_k(t) - y_{k-1}|$. Firstly, we will use the definition (3) of the Picard iterate, again reprinted here:

$$y_n(t) = y_0 + \int_{t_0}^{t} f(\tau, y_{n-1}(\tau))\, d\tau$$

In particular, we find that

$$\begin{aligned} |y_k(t) - y_{k-1}(t)| &= \left| \left(y_0 + \int_{t_0}^{t} f(\tau, y_{k-1}(\tau))\, d\tau \right) - \left(y_0 + \int_{t_0}^{t} f(\tau, y_{k-2}(\tau))\, d\tau \right) \right| \\ &= \left| \int_{t_0}^{t} f(\tau, y_{k-1}(\tau)) - f(\tau, y_{k-2}(\tau))\, d\tau \right| \\ &\le \int_{t_0}^{t} \Big| f(\tau, y_{k-1}(\tau)) - f(\tau, y_{k-2}(\tau)) \Big| d\tau, \end{aligned}$$

provided that $k \ge 2$. Next we invoke the mean value theorem, which states that if a function g is continuous on $[a, b]$ and differentiable on (a, b) then there exists a number $\xi \in (a, b)$ such that

$$g'(\xi) = \frac{g(b) - g(a)}{b - a}.$$

Now, for any given τ we can define

$$\begin{aligned} g(y) &= f(\tau, y) \\ a &= y_{k-2}(\tau) \\ b &= y_{k-1}(\tau). \end{aligned}$$

If we assume that f is continuous and the partial derivative $f_y = \partial f/\partial y$ exists, i.e. that g is differentiable, then the mean value theorem tells us that there exists a number ξ between $y_{k-2}(\tau)$ and $y_{k-1}(\tau)$ such that

$$f_y(\tau, \xi) = \frac{f(\tau, y_{k-1}(\tau)) - f(\tau, y_{k-2}(\tau))}{y_{k-1}(\tau) - y_{k-2}(\tau)}.$$

Rearranging this equation, we find the useful relation

$$f(\tau, y_{k-1}(\tau)) - f(\tau, y_{k-2}(\tau)) = f_y(\tau, \xi)\big[y_{k-1}(\tau) - y_{k-2}(\tau)\big].$$

If we make this argument for every $t_0 \le \tau \le t$, we may obtain a different number ξ for each τ. That is, we must replace the number ξ with a function $\xi(\tau)$. We then find that

$$\begin{aligned} \int_{t_0}^{t} \Big| f(\tau, y_{k-1}(\tau)) - f(\tau, y_{k-2}(\tau)) \Big| d\tau &= \int_{t_0}^{t} \Big| f_y(\tau, \xi(\tau))\big[y_{k-1}(\tau) - y_{k-2}(\tau)\big] \Big|\, d\tau \\ &= \int_{t_0}^{t} \Big| f_y(\tau, \xi(\tau)) \Big| \Big| y_{k-1}(\tau) - y_{k-2}(\tau) \Big|\, d\tau. \end{aligned}$$

Now, let us assume that $\partial f/\partial y$ not only exists over the rectangle R, but it is also continuous.[1] Then $|\partial f/\partial y|$ is also continuous, and therefore has a maximum value on R. We will denote this maximum value by L, i.e.

$$L = \max_{(t,y)\in R} |f_y(t,y)|.$$

Since $\xi(\tau)$ lies between the Picard iterates $y_{k-2}(\tau)$ and $y_{k-1}(\tau)$, our work in the previous section proves that it lies between the lines $y = y_0 \pm M(t-t_0)$ for $t_0 \le \tau \le \alpha$. Hence, all points $(\tau, \xi(\tau))$ lie within R for $t_0 \le \tau \le t$, and so $|f_y(\tau, \xi(\tau))| \le L$. The same elementary properties of definite integrals we used earlier apply again, so that

$$\int_{t_0}^{t} \Big|f_y(\tau,\xi(\tau))\Big| \Big|y_{k-1}(\tau) - y_{k-2}(\tau)\Big| \, d\tau \le L \int_{t_0}^{t} \Big|y_{k-1}(\tau) - y_{k-2}(\tau)\Big| \, d\tau.$$

In summary,

$$|y_k(t) - y_{k-1}(t)| \le L \int_{t_0}^{t} \Big|y_{k-1}(\tau) - y_{k-2}(\tau)\Big| \, d\tau$$

for every $k \ge 2$. We now switch to an inductive argument on k. For $k = 1$, recall we proved in the previous section that $|y_1(t) - y_0(t)| \le M(t-t_0)$ for $t_0 \le t \le t_0 + \alpha$. For $k = 2$, we have

$$\begin{aligned} |y_2(t) - y_1(t)| &\le L \int_{t_0}^{t} \Big|y_1(\tau) - y_0(\tau)\Big| \, d\tau \\ &\le L \int_{t_0}^{t} M(t-t_0) \, d\tau \\ &= \frac{ML(t-t_0)^2}{2}. \end{aligned}$$

For $k = 3$, we have

$$\begin{aligned} |y_3(t) - y_2(t)| &\le L \int_{t_0}^{t} \Big|y_2(\tau) - y_1(\tau)\Big| \, d\tau \\ &\le L \int_{t_0}^{t} \frac{ML(t-t_0)^2}{2} \, d\tau \\ &= \frac{ML^2(t-t_0)^3}{3!}. \end{aligned}$$

Inductively, we find that

$$|y_k(t) - y_{k-1}(t)| \le \frac{ML^{k-1}(t-t_0)^k}{k!} \tag{4}$$

for $t_0 \le t \le t_0 + \alpha$. But now we can easily show that the series $\sum_{k=1}^{\infty} |y_k(t) - y_{k-1}|$ converges, because

$$\begin{aligned} \sum_{k=1}^{\infty} \Big|y_k(t) - y_{k-1}(t)\Big| &\le \sum_{k=1}^{\infty} \frac{ML^{k-1}(t-t_0)^k}{k!} \\ &= \frac{M}{L} \sum_{k=1}^{\infty} \frac{[L(t-t_0)]^k}{k!} \\ &= \frac{M}{L} \left[\sum_{k=0}^{\infty} \left[\frac{[L(t-t_0)]^k}{k!} \right] - 1 \right] \end{aligned}$$

[1]This is not strictly necessary. All we need is that $|\partial f/\partial y|$ is bounded.

$$= \frac{M}{L}\left(e^{L(t-t_0)} - 1\right).$$

As $t_0 \le t \le t_0 + \alpha$, we have $t - t_0 \le \alpha$ and $e^{L(t-t_0)} \le e^{L\alpha}$. This shows that

$$\sum_{k=1}^{\infty} \Big| y_k(t) - y_{k-1}(t) \Big| \le \frac{M}{L}\left(e^{L\alpha} - 1\right),$$

which implies the series converges. This completes our proof that the sequence of Picard iterates $\{y_0(t), y_1(t), \ldots\}$ converges. We thus can define $y(t) = \lim_{n\to\infty} y_n(t)$.

7. Proof that $y(t)$ satisfies equation (2)

We will now show that the function $y(t) = \lim_{n\to\infty} y_n(t)$ satisfies equation (2), reprinted here:

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau.$$

To do so, we start with the definition (3) of the Picard iterates, reprinted here:

$$y_{n+1}(t) = y_0 + \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau.$$

Taking the limits of both sides as $n \to \infty$ gives us

$$y(t) = y_0 + \lim_{n\to\infty} \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau;$$

to show that $y(t)$ satisfies (2), we must demonstrate that

$$\int_{t_0}^{t} f(\tau, y(\tau))\, d\tau = \lim_{n\to\infty} \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau,$$

or equivalently that

$$\lim_{n\to\infty} \left| \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau - \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau \right| = 0.$$

We may now use roughly the same procedure that we used to show the convergence of the sequence $\{y_0(t), y_1(t), \ldots\}$. In particular:

$$\begin{aligned}\left| \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau - \int_{t_0}^{t} f(\tau, y_n(\tau))\, d\tau \right| &= \left| \int_{t_0}^{t} f(\tau, y(\tau)) - f(\tau, y_n(\tau))\, d\tau \right| \\ &\le \int_{t_0}^{t} \Big| f(\tau, y(\tau))\, d\tau - f(\tau, y_n(\tau)) \Big|\, d\tau.\end{aligned}$$

Next, applying the mean value theorem shows that for every τ between t_0 and t, there is a number $\xi(\tau)$ between $y_n(\tau)$ and $y(\tau)$ such that

$$f_y(\tau, \xi(\tau)) = \frac{f(\tau, y(\tau)) - f(\tau, y_n(\tau))}{y(\tau) - y_n(\tau)},$$

or equivalently

$$f(\tau, y(\tau)) - f(\tau, y_n(\tau)) = f_y(\tau, \xi(\tau))\Big[y(\tau) - y_n(\tau) \Big].$$

We then find that

$$\int_{t_0}^{t} \Big| f(\tau, y(\tau)) - f(\tau, y_n(\tau)) \Big|\, d\tau = \int_{t_0}^{t} \Big| f_y(\tau, \xi(\tau)) \Big| \Big| y(\tau) - y_n(\tau) \Big|\, d\tau.$$

As $y(t)$ is the limit of a sequence of functions $y_n(t)$ which all lie within the closed rectangle R for $t_0 \le \tau \le t$, it follows that $y(t)$ also lies within R on that interval. Because $\xi(\tau)$ is between $y_n(\tau)$ and

$y(\tau)$, all points $(\tau, \xi(\tau))$ lie within R for $t_0 \le \tau \le t$, and $|f_y(\tau, \xi(\tau))| \le L$. We can thus conclude that

$$\int_{t_0}^{t} \Big|f_y(\tau,\xi(\tau))\Big|\Big|y(\tau) - y_n(\tau)\Big|\, d\tau \le L \int_{t_0}^{t} \Big|y(\tau) - y_n(\tau)\Big|\, d\tau.$$

Now observe that the relations

$$y_n(\tau) = y_0(\tau) + \sum_{k=1}^{n} \Big[y_k(\tau) - y_{k-1}(\tau)\Big]$$

and

$$y(\tau) = y_0(\tau) + \sum_{k=1}^{\infty} \Big[y_k(\tau) - y_{k-1}(\tau)\Big]$$

may be combined to obtain

$$y(\tau) - y_n(\tau) = \sum_{k=n+1}^{\infty} \Big[y_k(\tau) - y_{k-1}(\tau)\Big].$$

Also, relation (4) tells us that

$$|y_k(\tau) - y_{k-1}(\tau)| \le \frac{ML^{k-1}(\tau - t_0)^k}{k!},$$

so

$$\Big|y(\tau) - y_n(\tau)\Big| = \Bigg|\sum_{k=n+1}^{\infty} \Big[y_k(\tau) - y_{k-1}(\tau)\Big]\Bigg| \le \sum_{k=n+1}^{\infty} \Big|y_k(\tau) - y_{k-1}(\tau)\Big| \le \sum_{k=n+1}^{\infty} \frac{ML^{k-1}(\tau - t_0)^k}{k!},$$

and using the fact that $t_0 \le \tau \le t \le \alpha$ gives

$$\Big|y(\tau) - y_n(\tau)\Big| \le \sum_{k=n+1}^{\infty} \frac{ML^{k-1}\alpha^k}{k!} \tag{5}$$

Substituting (5) yields:

$$\int_{t_0}^{t} f_y(\tau,\xi(\tau))\Big[y(\tau) - y_n(\tau)\Big]\, d\tau \le L \int_{t_0}^{t} \sum_{k=n+1}^{\infty} \frac{ML^{k-1}\alpha^k}{k!}\, d\tau.$$

Since every term of this series is nonnegative, Tonelli's theorem guarantees that we may swap the integral and summation:

$$L \int_{t_0}^{t} \sum_{k=n+1}^{\infty} \frac{ML^{k-1}\alpha^k}{k!}\, d\tau = L \sum_{k=n+1}^{\infty} \int_{t_0}^{t} \frac{ML^{k-1}\alpha^k}{k!}\, d\tau;$$

this allows us to simplify as follows:

$$L \sum_{k=n+1}^{\infty} \int_{t_0}^{t} \frac{ML^{k-1}\alpha^k}{k!}\, d\tau \le M \sum_{k=n+1}^{\infty} \left[\frac{L^k\alpha^k}{k!}(t - t_0)\right] \le M\alpha \sum_{k=n+1}^{\infty} \frac{(L\alpha)^k}{k!}.$$

Since the latter summation is the tail end of a series expansion for $e^{L\alpha}$, it approaches zero as $n \to \infty$. To prove this formally, observe that

$$\sum_{k=n+1}^{\infty} \frac{(L\alpha)^k}{k!} = \sum_{1}^{\infty} \frac{(L\alpha)^k}{k!} - \sum_{1}^{n} \frac{(L\alpha)^k}{k!} = e^{L\alpha} - \sum_{1}^{n} \frac{(L\alpha)^k}{k!}$$

and

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\frac{(L\alpha)^k}{k!}=e^{L\alpha}-\lim_{n\to\infty}\sum_{1}^{n}\frac{(L\alpha)^k}{k!}=0.$$

Since

$$\left|\int_{t_0}^{t} f(\tau,y(\tau))\,d\tau-\int_{t_0}^{t} f(\tau,y_n(\tau))\,d\tau\right|\le M\alpha\sum_{k=n+1}^{\infty}\frac{(L\alpha)^k}{k!}$$

and

$$\lim_{n\to\infty}\sum_{k=n+1}^{\infty}\frac{(L\alpha)^k}{k!}=0,$$

it follows that

$$\lim_{n\to\infty}\left|\int_{t_0}^{t} f(\tau,y(\tau))\,d\tau-\int_{t_0}^{t} f(\tau,y_n(\tau))\,d\tau\right|=0,$$

which completes the proof that $y(t)=\lim_{n\to\infty}y_n(t)$ satisfies (2).

8. Proof that $y(t)$ is continuous

To show that $y(t)$ is continuous, we must show that for every $\epsilon>0$ there exists a $\delta>0$ such that $|h|<\delta$ implies $|y(t+h)-y(t)|<\epsilon$. To do so, we observe that

$$y(t+h)-y(t)=[y(t+h)-y_n(t+h)]+[y_n(t+h)-y_n(t)]+[y_n(t)-y(t)]$$

and consequently

$$|y(t+h)-y(t)|\le|y(t+h)-y_n(t+h)|+|y_n(t+h)-y_n(t)|+|y_n(t)-y(t)|$$

for every $n\ge0$. By picking a large enough n, we can reduce the magnitude of this sum to ϵ. Since the summation

$$\sum_{k=n+1}^{\infty}\frac{ML^{k-1}\alpha^k}{k!}$$

is the tail end of a convergent Maclaurin series, we can make it as small as we wish by selecting a sufficiently large n. In particular, we will choose an n such that

$$\sum_{k=n+1}^{\infty}\frac{ML^{k-1}\alpha^k}{k!}<\frac{\epsilon}{3}$$

and relation (5) implies

$$\Big|y(\tau)-y_n(\tau)\Big|<\frac{\epsilon}{3},$$

for both $\tau=t$ and $\tau=t+h$.

This takes care of two out of the three terms. For the third, $|y_n(t+h)-y_n(t)|$, note that $y_n(t)$ is continuous for every $n\ge0$, and so for every $\epsilon'>0$ there exists a $\delta'>0$ such that $|h|<\delta'$ implies $|y_n(t+h)-y_n(t)|<\epsilon'$. We let $\epsilon'=\epsilon/3$ and define $\delta=\delta'$. Thus, each of the three terms is strictly less than $\epsilon/3$, and

$$|y(t+h)-y(t)|<\frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.$$

This completes the proof that $y(t)$ is continuous.

9. Proof that the solution to equation (2) is unique

Having shown that a solution to (1) exists, we now show that it is unique. Supposing that two solutions of (1) are given by $y(t)$ and $z(t)$, we define $w(t) = |y(t) - z(t)|$; it is sufficient, then, to show that $w(t) = 0$ for all t.

If $y(t)$ and $z(t)$ are solutions of (1), then they are also solutions of (2); that is,

$$y(t) = y_0 + \int_{t_0}^{t} f(\tau, y(\tau))\, d\tau$$
$$z(t) = y_0 + \int_{t_0}^{t} f(\tau, z(\tau))\, d\tau.$$

Subtracting, we find that

$$\big|y(t) - z(t)\big| = \left|\int_{t_0}^{t} f(\tau, y(\tau))\, d\tau - \int_{t_0}^{t} f(\tau, z(\tau))\, d\tau\right|$$
$$w(t) \le \int_{t_0}^{t} \Big|f(\tau, y(\tau)) - f(\tau, z(\tau))\Big|\, d\tau,$$

and by the same reasoning with the mean value theorem that we used twice before,

$$w(t) \le L \int_{t_0}^{t} \big|y(\tau) - z(\tau)\big|\, d\tau = L \int_{t_0}^{t} w(\tau)\, d\tau.$$

We now define

$$U(t) = \int_{t_0}^{t} w(\tau)\, d\tau,$$

so that

$$U'(t) = w(t) \le L \int_{t_0}^{t} w(\tau)\, d\tau = LU(t),$$

or equivalently

$$U'(t) - LU(t) \le 0.$$

Multiplying both sides by the strictly positive integrating factor $e^{-L(t-t_0)}$ gives

$$U'(t)e^{-L(t-t_0)} - LU(t)e^{-L(t-t_0)} \le 0$$
$$\frac{d}{dt}\Big[U(t)e^{-L(t-t_0)}\Big] \le 0.$$

Now, $U(t_0) = \int_{t_0}^{t_0} w(\tau)\, d\tau = 0$, so the function $U(t)e^{-L(t-t_0)}$ is zero at $t = t_0$. Furthermore, $w(t)$ is nonnegative and $t \ge t_0$, so $U(t)$ is also nonnegative. Since the function $U(t)e^{-L(t-t_0)}$ is zero at $t = t_0$, is never less than zero, and is never increasing, it must be zero for all $t > t_0$ as well. We must then conclude that $U(t) = 0$ and therefore that $w(t) = 0$. This completes the proof.